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Methods for Determining Stress Distribution in Adherends and Adhesives

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Introduction

For a number of years, considerable testing has been performed on many structural adhesives supplied by various adhesives manufacturers. The majority of these tests were, specifically, lap shear tests, peel tests, and creep tests which, more or less, conformed to the ASTM standards for testing materials. When performed properly, these are good tests—for quality control of a product or the comparison of one adhesive with another—but in the realm of designing a good structural adhesive bonded joint, the information thus obtained leaves much to be desired.

To effectively design a good bonded joint, not only the physical properties of the bonding agent but also the stress distribution in and around the joint becomes of practical importance. Knowing the stress distribution allows the designer to confidently set up design limits, to take advantage of the serviceability, elimination of local stress concentrations, and weight saving inherent in adhesive bonded structures.

When a lap joint is loaded in tension, the stress in the adherends increases over the nominal due to the offset of the joint. The offset produces a bending of the adherend, and this bending stress is in addition, or additive, to the original tensile stress. The determination of the stress distribution in the adherend then is dependent upon the tensile stress, or load divided by the area, plus the stress produced by bending.

For the simplest analysis of the bending stress, a lap joint as in Figure 1 was used with adherends of equal thickness and like material. The load is applied at points L and R on the adherends, equidistant from the center of the bond, permitting the treatment of only one side due to the symmetry about the center of the bond.

Fig. 1. Unloaded lap joint.

Upon loading the system, a deflection will occur which reduces the lever arm of the force acting at each element by the amount of this deflection, Figure 2.

If only very small deflections are considered, the maximum of the bending moment approaches Pt. Thus, higher external loads will decrease this amount. The unaffected tensile stress at some distance from the joint is P/A. The bending stress will then increase or decrease the tensile stress according to the law of superposition.

Assuming that the loaded adherends act like cylindrically bent plates[1] the deflection can be expressed by $d^2y/dx^2 = -M/D$ where D is the flexural rigidity and $D = [(Eh^3)/12\ (1 - \nu^2)]$.

The general solution of this second-order differential equation, with suitable boundary conditions, will give the bending deflection of the adherend. This deflection times the load produces the moment, and when divided by the section modulus gives the maximum bending stress. By the law of superposition, the bending stress added to the tensile stress gives the maximum stress.

The solution of this equation will not be made here[3]; however, the general equation for the stress distribution becomes:

$$\sigma = \sigma^t\left(1 + \frac{P}{Q}\right)$$

where:
$$P = 3(e^{k_1 f} + e^{-k_1 f})(1 + q)e^{-k_2 x_2}$$
$$Q = (\sqrt{(2+q)^3} + 1)e^{k_1 f} - (\sqrt{(2+q)^3} - 1)e^{-k_1 f} \qquad (1)$$

and σ^t is P/bh_2, E_2 is the modulus of elasticity of the adherend, g is the glue line thickness, h_2 is the adherend thickness, ν is the Poisson's ratio, x_2 is the location on adherend, f is $^1/_2$ the overlap, b is the width of adherend,

$$k_1 = \sqrt{\frac{\sigma^t\ 12(1-\nu^2)}{E_2\,(2+g/h_2)^3}}\ \frac{1}{h_2}$$

$$k_2 = \sqrt{\frac{\sigma^t\ 12(1-\nu^2)}{E_2}}\ \frac{1}{h_2}$$

and q is the ratio g/h_2.

Realizing that one of the prime advantages in bonding is to join unlike materials, the expression of the stress distribution in the adherends was expanded to include adherends of unlike material and unequal thickness. The expression thus obtained became:

$$\sigma_1 = \sigma^{t_1}\left(1 \pm \frac{R}{S}\right)$$

where:

$$R = \left\{\frac{3}{2}\left(\frac{n+1+2q-\beta}{n}\right) \times \left[(\sqrt{\rho}+1)e^{2k_4f} + (\sqrt{\rho}-1)e^{-2k_4f}\right] + 3\sqrt{\rho}\left(\frac{n+1+2q+\beta}{n}\right)\right\} e^{-k_1x_1}$$

$$S = \left(\sqrt{\rho\frac{E_2}{E_1n^3}}+1\right)(\sqrt{\rho}+1)e^{2k_4f} - \left(\sqrt{\rho\frac{E_2}{E_1n^3}}-1\right)(\sqrt{\rho}-1)e^{-2k_4f} \qquad (2)$$

$$\sigma_2 = \sigma^{t_2}\left(1 \pm \frac{T}{U}\right)$$

where:

$$T = \left\{\frac{3}{2}(n+1+2q+\beta) + \left[\left(\sqrt{\rho\frac{E_2}{E_1n^3}}+1\right)e^{2k_4} + \left(\sqrt{\rho\frac{E_2}{E_1n^3}}-1\right)e^{-2k_4f}\right] + 3\sqrt{\rho\frac{E_2}{E_1n^3}} \times (n+1+2q-\beta)\right\} e^{-k_2x_2}$$

$$U = \left(\sqrt{\rho\frac{E_2}{E_1n^3}}+1\right)(\sqrt{\rho}+1)e^{2k_4f} - \left(\sqrt{\rho\frac{E_2}{E_1n^3}}-1\right)(\sqrt{\rho}-1)e^{-2k_4f} \qquad (3)$$

and $\sigma_{1,2}$ is the total maximum stress in the adherends; $\sigma^{t_{1,2}}$ is the nominal tensile stress in the adherends; $h_{1,2}$ is the thickness of adherend; b is the width of plate; $E_{1,2}$ is the modulus of elasticity, adherends; E_3 is the modulus of elasticity, adhesive; g is the glue line thickness; n is the ratio h_1/h_2; q is the ratio g/h_2; β is $3n + 1 + 2q - 4\alpha$; and

$$\alpha = \frac{n + \frac{1}{2} + q + qn(E^3/E_2) + n^2E_1/2E_2}{n(E_1/E_2) + 1 + q(E^3/E_2)}$$

$$\rho = E_1/E_2\left[n^3 + 12n\left(\alpha - \frac{n}{2}\right)^2\right] + \frac{E^3}{E_2}\left[q^3 + 12q\left(n + \frac{q}{2} - \alpha\right)^2\right] + 12\left(n + \frac{1}{2} + q - \alpha\right)^2 + 1$$

$$k_1 = k_2\sqrt{E_2/E_1n^3}$$

$$k_2 = \sqrt{\frac{\sigma^{t_2}\,12(1-\nu^2)}{E_2\,h_2^2}}$$

$$2k_4f = \sqrt{\frac{\sigma^{t_2}\,12(1-\nu^2)}{E_2\rho}}\ \frac{2f}{h_2}$$

where $2f$ is the length of overlap, $x_{1,2}$ is the distance from edge of joint in the adherends, and ν is Poisson's ratio.

The plus or minus sign before the fraction determines the side of the adherend being analyzed. The plus sign would be the side in tension and the minus sign would be the side in compression. At low tensile stresses it is possible for the bending stress on the compression side to be greater than the tensile stress, producing a negative stress.

The majority of practical joint designs exhibit a condition of no external bending moment. This type of joint is evidenced in the edge design of sandwich panels and plates attached to beams. This type of configuration is usually preferred due to the complexity of the single lap joint.

Even though the analysis for the deflection and stresses in the adherends and stresses in the glue

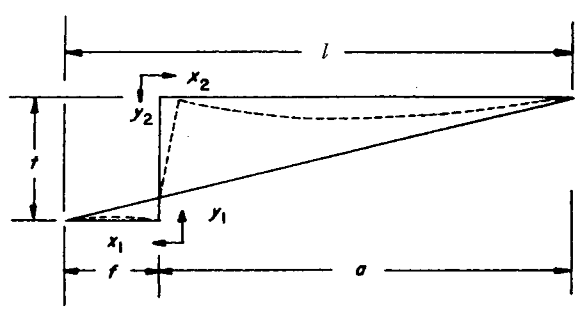

Fig. 2. Neutral plane undeflected (—) and deflected (—); choice of coordinate system.

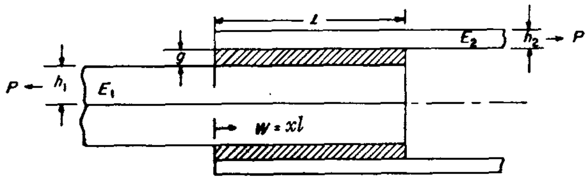

Fig. 3. Double overlap and notations for analysis.

line[2] have been made for single lap joints, the deflection of the adherends caused by the offset of the neutral planes from the line of acting force in many cases cannot be tolerated.

A double lap joint typifies the condition of no bending in the adherends (see Fig. 3). For stress distribution analyses the assumptions must be made that the deformation of the adherends is due solely to the longitudinal stress parallel to the adherend, the shear stress does not vary across the glue line, the tensile stresses parallel to the glue line are negligible, and the shear stress-shear strain relationship of the adhesive is constant. Of these assumptions all are very logical except perhaps the shear stress-shear strain relationship of the adhesive which is true only in particular cases.

Since a double lap shear specimen presents a symmetrical configuration it is necessary to analyze the stress distribution of only one bond. Designating the thick adherend as h_1, the thin bonded adherend as h_2, and the length of overlap as l with x incrementing from 0 to 1 along the overlap length, the stress function for an elemental portion of the adhesives can be expressed as $P_2(x) = b_2 l \int_0^x \tau(x)\, dx$, and the strain function $P_2(x) = b_2 l G \int_0^x \gamma(x) dx$. The total shear deflection of the adhesive at any point along the glue line is composed of the difference in tensile strain between the adjacent adherends plus the shear deflection of the adhesive at $x = 0$.

The strain in adherend 2 is

$$\epsilon_2(x) = \frac{P_2(x)}{E_2 b_2 h_2} = \frac{b_2 l G}{E_2 b_2 h_2} \int_0^x \gamma(x)\, dx$$

and the strain in adherend 1 is

$$\epsilon_1(x) = \frac{P - P(x)}{E_1 b_1 h_1} = \frac{1}{E_1 b_1 h_1}\left(\tau_m b l - b_2 l G \int_0^x \gamma(x) dx\right)$$

where τ_m = mean shear stress according to $P = \tau_m b_2 l$.

Incorporating the tensile strain functions of the adherends in the shear strain equation of the adhesive and differentiating twice leads to the differential equation describing the shear strain in the adhesive layer

$$\frac{d^2\gamma(x)}{dx^2} = k^2\gamma(x)$$

where k^2 is a constant and

$$k^2 = \frac{l^2 G\ (E_1 h_1 + E_2 h_2)}{g E_1 h_1 E_2 h_2}$$

Solving the second-order differential equation of the shear strain in the adhesives and the constants of integration produces

$$\gamma(x) = \gamma_m \sqrt{\frac{l^2 G E_2 h_2}{g E_1 h_1\ (E_1 h_1 + E_2 h_2)}} \times \left[\frac{\frac{E_1 h_1 + E_2 h_2}{E_2 h_2} \cosh kx + \cosh k\,(1 - x) - \cosh kx}{\sinh k}\right] \quad (4)$$

When the ratio of stresses and strains are assumed to be proportional, the ratio of the strains is equal to the ratio of the stresses and

$$\tau(x) = \tau_m \sqrt{\frac{l^2 G E_2 h_2}{g E_1 h_1 (E_1 h_1 + E_2 h_2)}} \times \left[\frac{\frac{E_1 h_1}{E_2 h_2} \cosh kx + \cosh k\,(1 - x)}{\sinh k}\right]$$

The tensile stress of adherend 2 at location x is $\sigma(x) = P(x)/bh_2$; incorporating this in the shear stress of the adhesive produces the stress distribution expression for the adherend:

$$\sigma_2\,(x) = \sigma_0 \frac{E_2 h_2}{E_1 h_1 + E_2 h_2} \times \left[\frac{\frac{E_1 h_1}{E_2 h_2} \sinh kx - \sinh k(1 - x)}{\sinh k} + 1\right]$$

where $\sigma_2(x)$ is the tensile stress in adherend 2 at location x, σ_0 is the tensile stress in adherend outside the joint, $h_{1,2}$ is the adherend thickness, $E_{1,2}$ is the modulus of elasticity of adherends, x is the amount of $0 < x < 1$,

$$k = \sqrt{\frac{l^2 G (E_1 h_1 + E_2 h_2)}{g E_1 h_1 E_2 h_2}}$$

where G is the shear modulus of adhesive.

Checking for the stresses at the beginning and the end of the overlap, which is $x = 0$ and $x = 1$ yields

$$\sigma\ (x) = 0\ x = 0$$

$$\sigma\ (x) = \sigma_0\ x = 1$$

The above analysis provides the equation for the stress distribution in the adhesive layer of a lap joint. The adhesive is assumed to be of linear elasticity. The actual nonlinear shear stress-shear strain relationship of the adhesive, particularly at elevated temperatures, tends to make the analysis conservative since these conditions lead to a lower stress at the same amount of strain.

Experimental

These stress distribution expressions were programmed for a Royal McBee LGP-30 digital computer, using a 24.2 program routine (an interpretative, floating point programing system), so that by inserting data consisting of the adherend width, thickness, and modulus of elasticity, overlap of the joint, glue line thickness and modulus of elasticity, tensile load, and incrementing the locations, the print out will provide the maximum stress at the indicated locations.

The data input describes the joint configuration.

Having evolved the expressions for the stress distribution in the adherends, the next step was to prove their correctness experimentally. This was accomplished by means of the photo stress technique.

Photo stress is a photoelastic stress analysis technique in which the structure to be stress analyzed is coated with a special clear plastic that becomes doubly refractive when stressed. Under polarized light the overall distribution of surface strains appears as black and color fringes in the plastic.

The part, or structure, to be stress analyzed is coated with a thin layer of transparent photoelastic plastic. When a load is applied, strains are transmitted to the plastic coating which then becomes doubly refractive. This change, or birefringence, is directly proportional to the intensity of stress. If a reflective surface is provided on the structure, birefringence can be observed using polarized light in a reflection polariscope. When examined in the field of polarized light of the instrument, black and colored fringe patterns are seen which reveal the complete geography of mechanical strains in the structure and can be interpreted in terms of principal stress. When a ray of light enters a doubly refractive material at normal incidence, the plastic under stress splits the ray into two plane-polarized components. When the rays are 180° out of phase, colored or black lines appear which are known as isochromatic lines. When monochromatic light is used, these lines are black. When white light is used, the patterns appear colored.

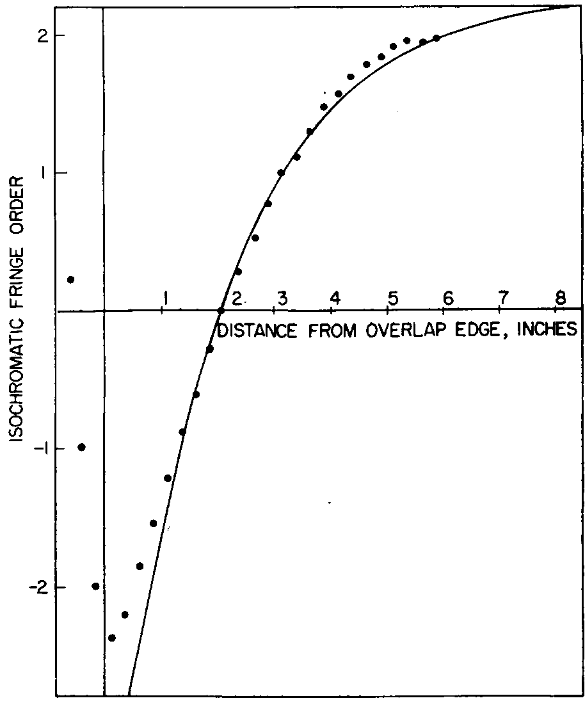

Fig. 4. Comparison of theory and experimental: (—) theory, (●) experimental.

A nonuniform strain yields variously colored striations and patches which, on analysis, indicate the magnitude of difference between the principal stresses. The magnitude of the principal stresses controls the degree of birefringence and, therefore, the extent to which the polarized rays are out of phase.

In line with the photo stress technique, a number of lap shear specimens of various configurations were made, and isochromatic readings taken. These readings were plotted and compared to the theory (Fig. 4). The accuracy of the isochromatic fringe constant is of the order of 2%. The error of the test readings above 0.5 fringes on the tested object is less than 3%.

Within experimental error the test results proved the theoretical expression to be correct (Fig. 4).

The above method for determining the stress distribution is not limited to adhesive bonded joints, but can be applied to soldered, welded, brazed, or in some cases, riveted joints as well.

References

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Synopsis

Loading a lap joint in tension causes the adherends to increase their tensile stress over the nominal because of the bending produced by the offset of the joint. Knowing the configuration of the joint and the tensile stress, the stress distribution in the adherends outside the joint can be determined. The theory, with resulting equations, is based on the assumption that the adherends act like a plate bent to a cylindrical surface. If the bending of the adherends is eliminated by the use of a double overlap joint, the stress distribution in the adhesive can be determined. This analysis is based on existing mathematical findings of the load distribution in rivets of riveted joints. The resulting equations have been proven experimentally by the use of the photo stress technique, and can be applied to adhesively bonded, soldered, brazed, welded, and under certain conditions, riveted joints.

Résumé

La charge d'un joint rodé sous tension, force les surfaces adhérentes à augmenter leur force de tension au-dessus de leur valeur apparente à cause de la courbure produite par le retrait de ce dernier. En connaissant la configuration du joint et la force de tension, on peut déterminer la distribution de tension dans les surfaces adhérentes à l'extérieur du joint. On base la théorie, et les équations qui en résultent, sur l'hypothèse que les surfaces adhérentes agissent comme des plaques courbées sur une surface cylindrique. Si la courbure des surfaces adhérentes est supprimée, par emploi d'un joint de recouvrement double, on peut déterminer la distribution de tension dans l'adhésif. On base cette analyse sur l'expression mathématique existante de la distribution de charge dans les rivets des joints. On a prouvé expérimentalement les équations résultants grâce à la technique de phototension, et on peut les appliquer aux joints liés par adhésion, soudés, brasés, soudés et dans certaines conditions aux joints rivés.

Zusammenfassung

Die Belastung einer Überlappungsverbindung unter Spannung verursacht wegen der durch die Versetzung in der Verbindung erzeugten Krümmung ein Ansteigen der Zugspannung der adhärierenden Stoffe über den Nominalwert. Bei Kenntnis des Aufbaus der Verbindung und der Zugspannung kann die Spannungsverteilung in den adhärierenden Stoffen ausserhalb der Verbindung bestimmt werden. Die Theorie und die daraus folgenden Gleichungen beruhen auf der Annahme, dass sich die adhärierenden Stoffe wie eine zu einer Zylinderoberfläche verbogene Platte verhalten. Wenn die Krümmung der adhärierenden Stoffe durch Verwendung einer doppelten Überlappungsverbindung beseitigt wird, kann die Spannungsverteilung im Klebestoff bestimmt werden. Die Analyse beruht auf bekannten mathematischen Befunden über die Belastungsverteilung in Nieten von Nietverbindungen. Die erhaltenen Gleichungen werden experimentell mittels des Photo-Spannungsverfahrens bestätigt und können auf geklebte, gelötete, geschmiedete, geschweisste und unter gewissen Bedingungen, auf genietete Verbindungen angewendet werden.